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2005 J. Phys. A: Math. Gen. 38 2259

(http://iopscience.iop.org/0305-4470/38/10/014)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 2259–2276 doi:10.1088/0305-4470/38/10/014

Heat kernel coefficients for chiral bag boundary conditions

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Received 7 October 2004, in final form 21 January 2005
Published 23 February 2005
Online at stacks.iop.org/JPhysA/38/2259

Abstract

We study the asymptotic expansion of the smeared L^2-trace of $f\,\mathrm{e}^{-tP^2}$ where P is an operator of Dirac type, f is an auxiliary smooth smearing function which is used to localize the problem, and chiral bag boundary conditions are imposed. Special case calculations, functorial methods and the theory of ζ- and η-invariants, are used to obtain the boundary part of the heat-kernel coefficients a_1 and a_2.

PACS number: 11.10.−z
Mathematics Subject Classification: 58J50

1. Introduction

Local boundary conditions for operators of Dirac type have been studied in the physical and mathematical literature with a variety of motivations over many years. Some key points in this respect are as follows.

(i) Local boundary conditions for massless fermionic fields ruled by a Dirac operator can be applied to one-loop quantum cosmology [12, 18] and are part of the investigation of conformal anomalies [29] in Euclidean field theory [19]. Moreover, they are the first step towards analysing boundary counterterms in supergravity theories, with the associated unsettled issue of proving finiteness [13] or lack of finiteness [14] of supergravity theories on manifolds with boundary. In other words, the local boundary conditions for fermionic fields are part of a general scheme [28] leading to locally supersymmetric boundary conditions for fermionic and bosonic fields [12, 3], and hence can be used to test perturbative consistency of supergravity models in cosmological [18, 20] or field-theoretical backgrounds.

(ii) Local boundary conditions of chiral bag type are a substitute for introducing small quark masses to drive the breaking of chiral symmetry [30]. One of the first papers where the chiral boundary conditions were introduced is the work by Hrasko and Balog [26], and one of the first applications to chiral bag models is presented in [24].

(iii) Chiral bag boundary conditions have been recently proved to lead to a strongly elliptic boundary-value problem for the squared Dirac operator [5], and the associated global heat-kernel asymptotics has been investigated in detail, on the Euclidean ball, in [21]. An early paper on the role of boundary conditions for Dirac operators is in the framework of fermionic billiards [2], studied even earlier by Berry and Mondragon [7].

For more general Riemannian manifolds with boundary, however, the investigation of such a global asymptotics in the chiral bag case is, to our knowledge, an open research field, and it appears desirable to understand how far can one go by exploiting functorial methods (e.g. conformal rescalings of the metric) and special case calculations, which are tools frequently used in invariance theory [23, 27]. For this purpose, both algorithms are exploited in our paper, where the general mathematical setting is as follows.

Let $m = 2\overline{m}$ be even and let $P = \gamma_j \nabla_j + \psi$ be an operator of Dirac type on a compact oriented Riemannian manifold M of dimension m, where ∇ is a compatible unitary connection, i.e. $\nabla\gamma = 0$. The spinor space has then dimension $\mathrm{d}_s = 2^{\overline{m}}$, and the γ-matrices can be taken to be skew-adjoint and obeying the Clifford relation

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij}.$$

Near the boundary, let e_m be the inward-pointing unit normal and γ_m be the projection of the γ-matrices on e_m. Moreover, the generalization of γ_5 to an arbitrary even dimension is provided by

$$\widetilde{\gamma} \equiv i^{\overline{m}} \gamma_1 \ldots \gamma_m. \tag{1a}$$

The squared Dirac operator is studied with local boundary conditions of chiral bag type. These boundary conditions involve a real angle θ and they read

$$\Pi_- \varphi|_{\partial M} = 0, \tag{1b}$$

where we have introduced the 'projectors'

$$\Pi_\mp \equiv \tfrac{1}{2}(1 \pm e^{\theta\tilde{\gamma}} \tilde{\gamma} \gamma_m). \tag{1c}$$

Under the above assumptions, the squared operator P^2 is an operator $\widetilde{P}$ of Laplace type [23]. The associated heat kernel can be defined as the solution, for $t > 0$, of the heat equation

$$\left(\frac{\partial}{\partial t} + \widetilde{P}\right) U(x, x'; t) = 0, \tag{1d}$$

obeying the initial condition

$$\lim_{t \to 0} \int_M \mathrm{d}x' U(x, x'; t) \Psi(x') = \Psi(x), \tag{1e}$$

jointly with the boundary conditions $\mathcal{B}_\theta$ defined by

$$\Pi_- U(x, x'; t)|_{x \in \partial M} = 0, \qquad \Pi_- P_x U(x, x'; t)|_{x \in \partial M} = 0. \tag{1f}$$

Here, $\mathrm{d}x'$ denotes the Riemannian volume element of the manifold M and P_x denotes the Dirac operator with respect to the variable x. The L^2-trace of the heat semi-group is obtained by integrating the fibre trace Tr_V of the heat kernel diagonal $U(x,x;t)$ over M, and reads as

$$\mathrm{Tr}_{L^2}(\mathrm{e}^{-t\widetilde{P}}) = \int_M \mathrm{d}x\, \mathrm{Tr}_V U(x,x;t). \tag{1g}$$

In our paper, following [23], we are interested in a slight generalization of the previous equation, where $\mathrm{e}^{-t\widetilde{P}}$ is 'weighted' with a smooth scalar function f on M. More precisely, we are interested in the asymptotic expansion as $t \to 0^+$ of the functional trace

$$\mathrm{Tr}_{L^2}(f\,\mathrm{e}^{-t\widetilde{P}}) = \int_M \mathrm{d}x\, f(x)\, \mathrm{Tr}_V U(x,x;t). \tag{1h}$$

The results for the original problem are eventually recovered by setting $f = 1$, but it is crucial to keep f arbitrary throughout the set of calculations, as will be clear from the following sections.

The asymptotic expansion of such a functional trace has the form

$$\mathrm{Tr}_{L^2}(f\,\mathrm{e}^{-t\widetilde{P}}) \sim \sum_{n=0}^{\infty} t^{(n-m)/2} a_n(f, \widetilde{P}, \mathcal{B}_\theta). \tag{1i}$$

Note that there is a change of convention in the indexing of the Seeley coefficients with respect to the work in [21], i.e. our a_n is denoted therein by $a_{n/2}$. The coefficients $a_n(f, \widetilde{P}, \mathcal{B}_\theta)$ consist of two different parts, the interior part $a_n^M(f, \widetilde{P})$ and the boundary part $a_n^{\partial M}(f, \widetilde{P}, \mathcal{B}_\theta)$, i.e.

$$a_n(f, \widetilde{P}, \mathcal{B}_\theta) = a_n^M(f, \widetilde{P}) + a_n^{\partial M}(f, \widetilde{P}, \mathcal{B}_\theta). \tag{1j}$$

The interior parts $a_n^M(f, \widetilde{P})$ are obtained by integrating some geometric invariants (see below) over M and are independent of the boundary conditions. In contrast, the boundary parts $a_n^{\partial M}(f, \widetilde{P}, \mathcal{B}_\theta)$ are obtained by integrating some geometric invariants over the boundary ∂M and these parts depend in a crucial way on the boundary conditions. They will be the main concern of our research from now on. The interior invariants are built universally and polynomially from the metric tensor, its inverse, the Riemann curvature of M, the bundle curvature (if a vector bundle over M is given) and the endomorphism (or 'potential' term) in the squared operator P^2. By virtue of Weyl's work on the invariants of the orthogonal group, these polynomials can be formed by using only tensor products and contraction of tensor arguments. Here, the structure group is $O(m)$, m being the dimension of M. However, when a boundary occurs, the boundary structure group is $O(m-1)$, and the Weyl theorem is used again to construct invariants.

The structure of this paper is as follows. In section 2 we write down the general form of the leading coefficients a_1 and a_2. The special case calculation of [21] and different functorial techniques are used to determine part of the numerical multipliers of the geometric invariants. Further special cases are shown in section 3 and the complete a_1 and a_2 coefficients are determined. We end the paper with concluding remarks.

2. Determination of the leading coefficients

We first write down the general form of the leading two boundary contributions to the heat kernel (hereafter, L_{aa} is our notation for the trace of the extrinsic-curvature tensor of the boundary).

Lemma 2.1. *Let f be scalar. There exist universal constants $c_i(\theta, m)$ such that (hereafter our notation for the invariant integration measure on ∂M is simply* dy*)*

$$a_1^{\partial M}(f, \widetilde{P}, \mathcal{B}_\theta) = (4\pi)^{-(m-1)/2} \int_{\partial M} \mathrm{d}y\, \mathrm{Tr}_V(c_1(\theta, m) f), \tag{2a}$$

$$a_2^{\partial M}(f, \widetilde{P}, \mathcal{B}_\theta) = (4\pi)^{-m/2} \int_{\partial M} \mathrm{d}y \, \mathrm{Tr}_V (c_2(\theta, m) L_{aa} f + c_3(\theta, m) f \psi \tilde{\gamma} \gamma_m$$
$$+ c_4(\theta, m) f \psi \gamma_m + c_5(\theta, m) f \psi \tilde{\gamma} + c_6(\theta, m) f \psi + c_7(\theta, m) f_{;m}). \qquad (2b)$$

Proof. This is a direct consequence of the Weyl theorem on the invariants of the orthogonal group [23], as we said at the end of section 1. □

We next determine the universal multipliers $c_i(\theta, m)$, $i = 1, \ldots, 7$. We first exploit a known special case. As usual, the hypergeometric function is denoted by ${}_2F_1(a, b; c; z)$.

Lemma 2.2. *We have*

$$c_1(\theta, m) = \frac{1}{4}(\cosh^{m-1}\theta - 1), \qquad (2c)$$

$$c_2(\theta, m) = \frac{1}{2(m-1)} \left\{ \frac{2m-5}{3} + (2-m)\, {}_2F_1\left(1, \frac{m-1}{2}; \frac{3}{2}; \tanh^2\theta\right) \right\}. \qquad (2d)$$

Proof. In [21] the heat kernel coefficients for the given setting have been evaluated on the Euclidean ball for the case $\psi = 0$ and $f = 1$. The results obtained were

$$a_1 = \frac{\sqrt{\pi}\mathrm{d}_s}{2^m \Gamma(m/2)} (\cosh^{m-1}\theta - 1),$$
$$a_2 = \frac{(2m-5)\mathrm{d}_s}{3 \cdot 2^m \Gamma(m/2)} + \frac{\mathrm{d}_s}{2^m \Gamma(m/2)} \left\{ {}_2F_1\left(1, \frac{m-1}{2}; \frac{1}{2}; \tanh^2\theta\right) - (m-1) \right.$$
$$\left. \times\, {}_2F_1\left(1, \frac{m+1}{2}; \frac{3}{2}; \tanh^2\theta\right) \right\}.$$

The volume of the sphere, which is the boundary of the ball, is

$$\mathrm{vol}(S^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)}.$$

Using this to rewrite the coefficient a_1 proves assertion (2*c*), which agrees with equation (40) in [2] for the first boundary correction to the partition function.

To prove (2*d*) we first use the Gauss recursion formula, see e.g. [25], equation 9.137.17,

$$\gamma\, {}_2F_1(\alpha, \beta; \gamma; z) - (\gamma - \beta)\, {}_2F_1(\alpha, \beta; \gamma + 1; z) - \beta\, {}_2F_1(\alpha, \beta + 1; \gamma + 1; z) = 0,$$

to write a_2 for the m-ball as

$$a_2 = \frac{\mathrm{d}_s}{2^m \Gamma(m/2)} \left\{ \frac{2m-5}{3} + (2-m)\, {}_2F_1\left(1, \frac{m-1}{2}; \frac{3}{2}; \tanh^2\theta\right) \right\}.$$

Comparison with the general form (2*b*) then proves assertion (2*d*). Note that in the given setting, i.e. with $\psi = 0$ and $f = 1$, the $c_2(\theta, m) L_{aa} f$ term is the only term contributing. □

Remark 2.3. For $\theta = 0$ the boundary conditions reduce to standard boundary conditions of mixed type. For $\theta = 0$ we have

$$c_1(0, m) = 0,$$
$$c_2(0, m) = \frac{1}{2(m-1)} \left\{ \frac{2m-5}{3} + (2-m) \cdot 1 \right\} = \frac{1}{2(m-1)} \frac{1-m}{3} = -\frac{1}{6}.$$

To achieve comparison with the known results for mixed boundary conditions note that the auxiliary Hermitian endomorphism χ needed to define the splitting of the spinor bundle is [27]

$$\chi = -\tilde{\gamma} \gamma_m.$$

Let

$$\Pi_\pm = \tfrac{1}{2}(1 \pm \chi)$$

be the projection on the $\pm$ eigenspaces of χ. Mixed boundary conditions are then defined as

$$\mathcal{B}\varphi = \Pi_-\varphi|_{\partial M} \oplus (\nabla_m + S)\Pi_+\varphi|_{\partial M} = 0.$$

The relevant S for the given setting is

$$S = -\tfrac{1}{2}L_{aa}\Pi_+.$$

Using the fact that $\mathrm{Tr}_V(\chi) = 0$, $\mathrm{Tr}_V(\Pi_\pm) = \mathrm{d}_s/2$, the coefficients for the relevant mixed boundary conditions turn out to be:

$$a_1^{\partial M}(1, \widetilde{P}, \mathcal{B}_0) = 0,$$

$$a_2^{\partial M}(1, \widetilde{P}, \mathcal{B}_0) = (4\pi)^{-m/2}\frac{1}{6}\int_{\partial M} \mathrm{Tr}_V(2L_{aa} + 12S)\,\mathrm{d}y = (4\pi)^{-m/2}\frac{1}{6}\int_{\partial M} \mathrm{Tr}_V(-L_{aa})\,\mathrm{d}y,$$

in agreement with our findings for $c_1(0, m)$ and $c_2(0, m)$.

We next exploit the fact that the connection ∇ is not canonically defined. To simplify the notation slightly we assume a localizing function $f = 1$.

Lemma 2.4. *We have*

$$c_4(\theta, m) = 0.$$

Proof. Let σ_i be a skew-adjoint endomorphism of the spinor bundle commuting with the Clifford structure γ, $[\sigma_i, \gamma_j] = 0$. Then

$$\nabla_i(\epsilon) = \nabla_i + \epsilon\sigma_i$$

defines a smooth one-parameter family of compatible unitary connections. We define

$$\psi(\epsilon) := \psi - \epsilon\gamma_i\sigma_i$$

to ensure that

$$P(\epsilon) = \gamma_i\nabla_i(\epsilon) + \psi(\epsilon) = P$$

is unaffected by the perturbation; the boundary condition also remains unchanged. Therefore, the heat trace coefficient (2*b*) remains unchanged. Using $\tilde{\gamma}\gamma_i = -\gamma_i\tilde{\gamma}$ we evaluate the variation $\delta = (d/d\epsilon)|_{\epsilon=0}$ of the single terms for $\sigma_a = 0$, $\sigma_m \neq 0$:

$$\delta\,\mathrm{Tr}_V(c_2(\theta, m)L_{aa}) = 0,$$

$$\begin{aligned}\delta\,\mathrm{Tr}_V(c_3(\theta, m)\psi\tilde{\gamma}\gamma_m) &= -\mathrm{Tr}_V(c_3(\theta, m)\gamma_m\sigma_m\tilde{\gamma}\gamma_m) = \mathrm{Tr}_V(c_3(\theta, m)\sigma_m\tilde{\gamma})\\ &= -\mathrm{Tr}_V(c_3(\theta, m)\sigma_m\gamma_m\tilde{\gamma}\gamma_m) = -\mathrm{Tr}_V(c_3(\theta, m)\sigma_m\tilde{\gamma}) = 0,\end{aligned}$$

$$\delta\,\mathrm{Tr}_V(c_4(\theta, m)\psi\gamma_m) = -\mathrm{Tr}_V(c_4(\theta, m)\gamma_m\sigma_m\gamma_m) = \mathrm{Tr}_V(c_4(\theta, m)\sigma_m),$$

$$\begin{aligned}\delta\,\mathrm{Tr}_V(c_5(\theta, m)\psi\tilde{\gamma}) &= -\mathrm{Tr}_V(c_5(\theta, m)\gamma_m\sigma_m\tilde{\gamma}) = -\mathrm{Tr}_V(c_5(\theta, m)\sigma_m\tilde{\gamma})\\ &= \mathrm{Tr}_V(c_5(\theta, m)\sigma_m\gamma_m\tilde{\gamma}) = \mathrm{Tr}_V(c_5(\theta, m)\gamma_m\sigma_m\tilde{\gamma}) = 0,\end{aligned}$$

$$\begin{aligned}\delta\,\mathrm{Tr}_V(c_6(\theta, m)\psi) &= -\mathrm{Tr}_V(c_6(\theta, m)\gamma_m\sigma_m) = -\mathrm{Tr}_V(c_6(\theta, m)\gamma_m\sigma_m\tilde{\gamma}\tilde{\gamma})\\ &= \mathrm{Tr}_V(c_6(\theta, m)\tilde{\gamma}\gamma_m\sigma_m\tilde{\gamma}) = \mathrm{Tr}_V(c_6(\theta, m)\gamma_m\sigma_m) = 0.\end{aligned}$$

For the coefficient to remain unchanged we need $c_4(\theta, m) = 0$.

Considering $\sigma_a \neq 0$ and $\sigma_m = 0$ does not produce any new information. □

To find more information about the remaining unknown multipliers, one might enlarge the setting and allow for an endomorphism-valued f. However, apart from the fact that the number of invariants goes up to 36 and the calculation gets cumbersome, this does not produce any relevant information for our problem and we do not present further details.

Instead, we next exploit conformal rescaling techniques.

Lemma 2.5. *We have*

$$c_7(\theta, m) = -\frac{m-1}{m-2}\left(c_2(\theta, m) + \frac{1}{6}\right).$$

Proof. Let f be a smooth function with $f|_{\partial M} = 0$. Define $\mathrm{d}s^2(\epsilon) := e^{2\epsilon f}\mathrm{d}s^2$ and $P(\epsilon) := e^{-\epsilon f} P$. Let ∇ be a compatible unitary connection. We expand $P = \gamma^\nu \nabla_{\partial_\nu} + \psi$ with respect to a local coordinate system $x = (x_1, \ldots, x_m)$ and use the metric to lower indices and define γ_ν. If we define

$$\nabla(\epsilon)_{\partial_\mu} := \nabla_{\partial_\mu} + \tfrac{1}{2}\epsilon(f_{;\nu}\gamma^\nu\gamma_\mu + f_{;\mu}),$$

results of [16] show that $\nabla(\epsilon)$ is a compatible unitary connection. Furthermore,

$$\psi(\epsilon) = \mathrm{e}^{-\epsilon f}\left(\psi - \tfrac{1}{2}\epsilon(m-1) f_{;\nu}\gamma^\nu\right).$$

Note that the boundary condition remains unchanged under conformal variation. The heat kernel coefficients satisfy the equation

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} a_n(1, \widetilde{P}(\epsilon), \mathcal{B}_\theta) = (m-n)a_n(f, \widetilde{P}, \mathcal{B}_\theta). \tag{2e}$$

To study the numerical multiplier $c_7(\theta, m)$ we need the variations

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \tau(\epsilon) = -2f\tau - 2(m-1)\Delta f,$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} L_{aa}(\epsilon) = -f L_{aa} - (m-1) f_{;m},$$

where $\tau = R_{ijji}$ is the scalar curvature. Applying equation (2*e*) proves the assertion. □

Remark 2.6. Note that, despite the appearance, the multiplier $c_7(\theta, m)$ is well defined in dimension $m = 2$. Using the result for $c_2(\theta, m)$ given in equation (2*d*) we obtain explicitly

$$c_7(\theta, m) = -\frac{1}{2}\left\{1 - {}_2F_1\left(1, \frac{m-1}{2}; \frac{3}{2}; \tanh^2\theta\right)\right\}. \tag{2f}$$

For $\theta = 0$ this agrees with the previous computation for mixed boundary conditions.

3. Relating the zeta and eta invariants

In order to determine the numerical multipliers $c_3(\theta, m)$, $c_5(\theta, m)$ and $c_6(\theta, m)$ we relate the zeta invariant to the eta invariant. We will then evaluate the eta invariant on the m-dimensional cylinder and ball for the case of an endomorphism-valued f. On the ball we will restrict to the choices $f = 1$ and $f = \tilde{\gamma}$, respectively, which will allow us to find $c_5(\theta, m)$ and $c_6(\theta, m)$. Instead, on the cylinder we can deal with general f. Performing the two special case calculations is strictly speaking not necessary, but provides helpful crosschecks of the answers obtained.

To distinguish the coefficients in the heat trace, $\mathrm{Tr}_{L^2}\left(f\mathrm{e}^{-tP^2}\right)$, and in the trace related to the eta invariant, $\mathrm{Tr}_{L^2}\left(f P e^{-tP^2}\right)$, in this section we use the notation

$$\mathrm{Tr}_{L^2}\left(f\, \mathrm{e}^{-tP^2}\right) \sim \sum_n t^{(n-m)/2} a_n^\zeta(f, P^2, \mathcal{B}_\theta),$$

$$\mathrm{Tr}_{L^2}\left(f P\, \mathrm{e}^{-tP^2}\right) \sim \sum_n t^{(n-m-1)/2} a_n^\eta(f, P, \mathcal{B}_\theta).$$

The result we are going to need is the following:

Lemma 3.1. *Let $f \in C^\infty(End(V))$ and let $P_\epsilon := P + \epsilon f$. We then have*

$$\partial_\epsilon a_n^\zeta\left(1, P_\epsilon^2, \mathcal{B}_\theta\right) = -2a_{n-1}^\eta(f, P_\epsilon, \mathcal{B}_\theta).$$

Proof. The proof is insensitive to the boundary conditions imposed and parallels the proof in [10]. □

Remark 3.2. The very useful property of this result is that the a_n^ζ coefficient for the zeta invariant is related to the coefficient a_{n-1}^η for the eta invariant, which will have a significantly simpler structure.

In order to apply lemma 3.1 to the coefficient a_2^ζ we need the general form of the a_1^η coefficient.

Lemma 3.3. *Let $f \in C^\infty(End(V))$. There exist universal constants $d_i(\theta, m)$ such that*

$$a_1^{\eta,\partial M}(f, P, \mathcal{B}_\theta) = (4\pi)^{-m/2} \times \int_{\partial M} \mathrm{d}y \, \mathrm{Tr}_V\{d_1(\theta, m) f + d_2(\theta, m) f \tilde{\gamma} + d_3(\theta, m) f \gamma_m + d_4(\theta, m) f \tilde{\gamma} \gamma_m\}.$$

Proof. This follows immediately from the theory of invariants taking into account that f is in general a matrix-valued endomorphism. □

Remark 3.4. Lemma 3.1 relates the universal constant $d_j(\theta, m)$, $j = 1, \ldots, 4$, with $c_i(\theta, m)$, $i = 3, \ldots, 6$. In particular we have

$$c_3(\theta, m) = -2d_4(\theta, m), \qquad c_4(\theta, m) = -2d_3(\theta, m),$$
$$c_5(\theta, m) = -2d_2(\theta, m), \qquad c_6(\theta, m) = -2d_1(\theta, m).$$

From lemma 2.4 we conclude $d_3(\theta, m) = 0$. We evaluate $d_1(\theta, m)$ and $d_2(\theta, m)$ for the example of the ball and thus find $c_5(\theta, m)$ and $c_6(\theta, m)$. We also evaluate $d_1(\theta, m)$, $d_2(\theta, m)$ and $d_4(\theta, m)$ for the example of the cylinder. This provides checks of the answers for $c_5(\theta, m)$ and $c_6(\theta, m)$ and in addition determines $d_4(\theta, m)$ and thus $c_3(\theta, m)$.

For the case $f = 1$ we follow [22]. The case $f = \tilde{\gamma}$ is based upon this calculation and therefore we need to present some details for the case $f = 1$. We first summarize properties of the spectral resolution for the Dirac operator on the ball. Let $P = \gamma_i \nabla_i$ be the Dirac operator on the ball and let us denote by $\varphi_\pm$ its eigenfunctions obeying the eigenvalue equation $P\varphi_\pm = \pm\mu\varphi_\pm$. On writing the eigenvalue equation in this form we have $\mu > 0$. Later on we will write the eigenvalues of P as $\lambda = \pm\mu$, such that $|\lambda| = \mu$. Modulo a suitable radial normalizing constant C, we may express [15]

$$\varphi_{\pm}^{(+)} = \frac{C}{r^{(m-2)/2}} \begin{pmatrix} i J_{n+m/2}(\mu r) Z_{+}^{(n)}(\Omega) \\ \pm J_{n+m/2-1}(\mu r) Z_{+}^{(n)}(\Omega) \end{pmatrix} \tag{3a}$$

and

$$\varphi_{\pm}^{(-)} = \frac{C}{r^{(m-2)/2}} \begin{pmatrix} \pm J_{n+m/2-1}(\mu r) Z_{-}^{(n)}(\Omega) \\ i J_{n+m/2}(\mu r) Z_{-}^{(n)}(\Omega) \end{pmatrix}. \tag{3b}$$

Here, $J_\nu(z)$ are the Bessel functions and $Z_{\pm}^{(n)}(\Omega)$ are the eigenspinors of the Dirac operator $\check{P}$ on the sphere [11],

$$\check{P} \mathcal{Z}_{\pm}^{(n)}(\Omega) = \pm \left(n + \frac{m-1}{2} \right) \mathcal{Z}_{\pm}^{(n)}(\Omega) \text{ for } n = 0, 1, \ldots.$$

The degeneracy $d_n(m)$ for each eigenvalue is

$$d_n(m) := \dim \mathcal{Z}_{\pm}^{(n)}(\Omega) = \frac{1}{2} \mathrm{d}_s \begin{pmatrix} m+n-2 \\ n \end{pmatrix}.$$

We next apply the boundary operator which reads explicitly, from equation (1*b*),

$$\frac{1}{2} \begin{pmatrix} 1 & -\mathrm{i}\,\mathrm{e}^{\theta} \\ \mathrm{i}\,\mathrm{e}^{-\theta} & 1 \end{pmatrix},$$

to the solutions (3*a*) and (3*b*). This produces the following eigenvalue conditions:

$$J_{n+\frac{m}{2}}(\mu) \mp e^{\theta} J_{n+\frac{m}{2}-1}(\mu) = 0 \qquad \text{for} \quad \varphi_{\pm}^{(+)}, \tag{3c}$$

$$J_{n+\frac{m}{2}}(\mu) \pm e^{-\theta} J_{n+\frac{m}{2}-1}(\mu) = 0 \quad \text{for} \quad \varphi_{\pm}^{(-)}. \tag{3d}$$

These equations allow us to rewrite the eta function

$$\eta(s; 1, P, \mathcal{B}_\theta) = \sum_{\lambda} \mathrm{sgn}(\lambda) |\lambda|^{-s}$$

in terms of a contour integral and to apply the techniques described in detail in [6, 8, 9, 27]. The coefficients in the asymptotic expansion for the eta invariant are then determined by evaluating residues of η according to [23]

$$\mathrm{Res}\, \eta(m-n; 1, P, \mathcal{B}_\theta) = \frac{2a_n^{\eta}(1, P, \mathcal{B}_\theta)}{\Gamma\left(\frac{m-n+1}{2}\right)}. \tag{3e}$$

For notational convenience we introduce $p = n + m/2 - 1$. Starting point of the analysis is [22]

$$\eta(s; 1, P, \mathcal{B}_\theta) = \sum_{n=0}^{\infty} d_n(m) \frac{1}{2\pi i} \int_\Gamma \mathrm{d}k\, k^{-s} \frac{\mathrm{d}}{\mathrm{d}k} \ln \frac{1 + \mathrm{e}^{\theta} \frac{J_{p+1}(k)}{J_p(k)}}{1 - \mathrm{e}^{\theta} \frac{J_{p+1}(k)}{J_p(k)}} - (\theta \to -\theta),$$

where Γ is a suitable counterclockwise contour enclosing all solutions of the equations (3*c*) and (3*d*). After deforming the contour to the imaginary axis this gives

$$\begin{aligned}
\eta(s; 1, P, \mathcal{B}_\theta) &= \frac{1}{\pi i} \cos\left(\frac{\pi s}{2}\right) \sum_{n=0}^{\infty} d_n(m) \int_0^\infty \mathrm{d}z z^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{1 - \mathrm{i}\,\mathrm{e}^{\theta} \frac{I_{p+1}(z)}{I_p(z)}}{1 + \mathrm{i}\,\mathrm{e}^{\theta} \frac{I_{p+1}(z)}{I_p(z)}} - (\theta \to -\theta) \\
&= \frac{1}{\pi i} \cos\left(\frac{\pi s}{2}\right) \sum_{n=0}^{\infty} d_n(m) \int_0^\infty \mathrm{d}z z^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{1 + \frac{p}{z} \mathrm{i}\,\mathrm{e}^{\theta} - \mathrm{i}\,\mathrm{e}^{\theta} \frac{I_p'(z)}{I_p(z)}}{1 - \frac{p}{z} \mathrm{i}\,\mathrm{e}^{\theta} + \mathrm{i}\,\mathrm{e}^{\theta} \frac{I_p'(z)}{I_p(z)}} - (\theta \to -\theta),
\end{aligned}$$

where in the last step we have used the recursion for the modified Bessel function [25]

$$I_{\nu+1}(z) = I'_\nu(z) - \frac{\nu}{z} I_\nu(z). \tag{3f}$$

In order to recover the coefficient a_1^η we only need to consider the leading term in the uniform $p \to \infty$ asymptotic expansion of the Bessel function [1],

$$\frac{I'_p(kp)}{I_p(kp)} \sim \frac{(1+k^2)^{1/2}}{k}\left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right).$$

Hence we only need to find the residue of

$$A_0(s;1) = \frac{1}{\pi i}\cos\left(\frac{\pi s}{2}\right)\sum_{n=0}^\infty d_n(m) p^{-s}\int_0^\infty \mathrm{d}k\, k^{-s}\frac{\mathrm{d}}{\mathrm{d}k}\ln\frac{1+\frac{\mathrm{i}}{k}\mathrm{e}^\theta - \frac{\mathrm{i}\,\mathrm{e}^\theta\sqrt{1+k^2}}{k}}{1-\frac{\mathrm{i}}{k}\mathrm{e}^\theta + \frac{\mathrm{i}\,\mathrm{e}^\theta\sqrt{1+k^2}}{k}} - (\theta \to -\theta)$$

at $s = m-1$.

We first observe that the summation over n produces a multiple of the Barnes zeta function [4], which is defined by

$$\zeta_{\mathcal{B}}(s,a) := \sum_{n=0}^\infty \binom{m+n-2}{n}(n+a)^{-s}.$$

In detail we have

$$\sum_{n=0}^\infty d_n(m)p^{-s} = \frac{1}{2}\mathrm{d}_s\zeta_{\mathcal{B}}\left(s, \frac{m}{2}-1\right).$$

In order to perform the k-integral we first combine θ and $-\theta$ and evaluate the logarithmic derivative to give

$$\frac{\mathrm{d}}{\mathrm{d}k}\ln\frac{1+\frac{\mathrm{i}\,\mathrm{e}^\theta}{k}\left(1-\sqrt{1+k^2}\right)}{1-\frac{\mathrm{i}\,\mathrm{e}^\theta}{k}\left(1-\sqrt{1+k^2}\right)} - (\theta\to-\theta) = -\frac{4\mathrm{i}\sinh\theta}{\sqrt{1+k^2}(2+k^2+k^2\cosh(2\theta))}.$$

The relevant k-integral therefore reads

$$\begin{aligned}
&\int_0^\infty \mathrm{d}k k^{-s}(1+k^2)^{-3/2}\frac{1}{1+\frac{k^2}{2(1+k^2)}(\cosh(2\theta)-1)}\\
&\quad = \frac{1}{2}\Gamma\left(1+\frac{s}{2}\right)\sum_{l=0}^\infty(-1)^l\left(\frac{\cosh(2\theta)-1}{2}\right)^l\frac{\Gamma\left(\frac{1-s}{2}+l\right)}{\Gamma\left(\frac{3}{2}+l\right)}\\
&\quad = \frac{1}{\sqrt{\pi}}\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right){}_2F_1\left(1,\frac{1-s}{2};\frac{3}{2};\frac{1}{2}(1-\cosh(2\theta))\right)\\
&\quad = \frac{1}{\sqrt{\pi}}\Gamma\left(1+\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right){}_2F_1\left(1,\frac{1-s}{2};\frac{3}{2};-\sinh^2\theta\right).
\end{aligned}$$

From here, with [25]

$$\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\cos\left(\frac{\pi s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)},$$

we easily compute

$$A_0(s;1) = -\frac{1}{\sqrt{\pi}}\mathrm{d}_s\frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)}\sinh\theta\,{}_2F_1\left(1,\frac{1-s}{2};\frac{3}{2};-\sinh^2\theta\right)\zeta_{\mathcal{B}}\left(s,\frac{m}{2}-1\right).$$

Using [25]

$$\frac{\Gamma\left(\frac{m-1}{2}\right)}{\Gamma(m-1)} = \frac{\sqrt{\pi}}{2^{m-2}\Gamma\left(\frac{m}{2}\right)},$$

the coefficient a_1^η can be cast into the form

$$\begin{aligned} a_1^\eta(1, P, \mathcal{B}_\theta) &= \frac{1}{2}\Gamma\left(\frac{m}{2}\right)\text{Res } \eta(m-1; 1, P, \mathcal{B}_\theta) = \frac{1}{2}\Gamma\left(\frac{m}{2}\right)\text{Res } A_0(m-1; 1) \\ &= -\sinh\theta \mathrm{d}_s \frac{m-1}{2^m\Gamma\left(\frac{m}{2}\right)}\, {}_2F_1\left(1, 1-\frac{m}{2}; \frac{3}{2}; -\sinh^2\theta\right). \end{aligned}$$

Comparing this with the answer on the ball expected from lemma 3.3,

$$a_1^\eta(1, P, \mathcal{B}_\theta) = (4\pi)^{-m/2}\,\mathrm{vol}(S^{m-1})\mathrm{d}_s d_1(\theta, m) = \frac{2}{2^m\Gamma\left(\frac{m}{2}\right)}\mathrm{d}_s d_1(\theta, m),$$

we read off

$$d_1(\theta, m) = -\frac{m-1}{2}\sinh\theta\, {}_2F_1\left(1, 1-\frac{m}{2}; \frac{3}{2}; -\sinh^2\theta\right). \tag{3g}$$

From remark 3.4 we then get

$$c_6(\theta, m) = (m-1)\sinh\theta\, {}_2F_1\left(1, 1-\frac{m}{2}; \frac{3}{2}; -\sinh^2\theta\right).$$

To find the universal constant $c_5(\theta, m)$ we perform the calculation on the ball with $f = \tilde{\gamma}$. This choice complicates the analysis significantly because the normalization constant C and further integrals over products of Bessel functions come into play. First we note that if $\eta(s; x, y)$ denotes the local eta function, then

$$\eta(s; x, y) = \sum_\mu \mu^{-s}\left\{\varphi_+^{(\pm)}(x)^*\varphi_+^{(\pm)}(y) - \varphi_-^{(\pm)}(x)^*\varphi_-^{(\pm)}(y)\right\}.$$

We want to analyse

$$\mathrm{Tr}_{L^2}(\tilde{\gamma}\eta(s; x, x)) = \sum_\mu \mu^{-s}\left\{\left\langle\varphi_+^{(\pm)}\middle|\tilde{\gamma}\varphi_+^{(\pm)}\right\rangle - \left\langle\varphi_-^{(\pm)}\middle|\tilde{\gamma}\varphi_-^{(\pm)}\right\rangle\right\}, \tag{3h}$$

with $\langle\varphi_1|\varphi_2\rangle$ denoting the Hilbert space product

$$\langle\varphi_1|\varphi_2\rangle \equiv \int_M \mathrm{d}x\,\varphi_1^*(x)\varphi_2(x).$$

Since

$$\tilde{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

changes the sign of the lower chirality, the normalization constant C does not cancel in the Hilbert space products appearing in $(3h)$, but instead values of various integrals occur explicitly. We first observe that

$$\frac{1}{C^2} = \int_0^1 \mathrm{d}r\, r\left(J_{p+1}^2(\mu r) + J_p^2(\mu r)\right).$$

We use [25]

$$\int_0^1 \mathrm{d}r r J_\nu^2(\mu r) = \frac{1}{2}\left\{J_\nu^2(\mu) - J_{\nu-1}(\mu)J_{\nu+1}(\mu)\right\}$$

to find

$$\frac{1}{C^2} = \frac{1}{2}\left\{J_p^2(\mu) + J_{p+1}^2(\mu) - J_{p-1}(\mu)J_{p+1}(\mu) - J_p(\mu)J_{p+2}(\mu)\right\}.$$

We use the implicit eigenvalue equations (3c) and (3d) together with recursion relations for the Bessel functions [25]

$$J_{p+2}(\mu) = \frac{2(p+1)}{\mu}J_{p+1}(\mu) - J_p(\mu), \qquad J_{p-1}(\mu) = \frac{2p}{\mu}J_p(\mu) - J_{p+1}(\mu),$$

to simplify the normalization constants $C_\pm^{(\pm)}$ for the different spinors $\varphi_\pm^{(\pm)}$. We obtain

$$C_+^{(\pm)} = \frac{\sqrt{\mu}}{J_p(\mu)}\frac{1}{(\mu + \mu\,\mathrm{e}^{\pm 2\theta} \mp (2p+1)\,\mathrm{e}^{\pm\theta})^{1/2}},$$
$$C_-^{(\pm)} = \frac{\sqrt{\mu}}{J_p(\mu)}\frac{1}{(\mu + \mu\,\mathrm{e}^{\pm 2\theta} \pm (2p+1)\,\mathrm{e}^{\pm\theta})^{1/2}}.$$

Proceeding in the same way for the quantities $\langle\varphi_\pm^{(\pm)}|\tilde{\gamma}\varphi_\pm^{(\pm)}\rangle$, we find

$$\langle\varphi_+^{(\pm)}|\tilde{\gamma}\varphi_+^{(\pm)}\rangle = -\frac{\mathrm{e}^{\pm\theta}}{\mu + \mu\,\mathrm{e}^{\pm 2\theta} \mp (2p+1)\,\mathrm{e}^{\pm\theta}} = -\frac{1}{2\cosh\theta}\frac{1}{\mu \mp \frac{p+1/2}{\cosh\theta}},$$
$$\langle\varphi_-^{(\pm)}|\tilde{\gamma}\varphi_-^{(\pm)}\rangle = \frac{1}{2\cosh\theta}\frac{1}{\mu \pm \frac{p+1/2}{\cosh\theta}}.$$

Using these results in (3h), we obtain the following contour integral representation:

$$\eta(s;\tilde{\gamma},P,\mathcal{B}_\theta) = -\frac{1}{4\pi\mathrm{i}\cosh\theta}\sum_{n=0}^{\infty}d_n(m)\int_\Gamma \mathrm{d}k\,k^{-s}\frac{\frac{\mathrm{d}}{\mathrm{d}k}\ln[J_{p+1}(k) - \mathrm{e}^\theta J_p(k)]}{k - \frac{p+1/2}{\cosh\theta}}$$
$$-\frac{1}{4\pi\mathrm{i}\cosh\theta}\sum_{n=0}^{\infty}d_n(m)\int_\Gamma \mathrm{d}k\,k^{-s}\frac{\frac{\mathrm{d}}{\mathrm{d}k}\ln[J_{p+1}(k) + \mathrm{e}^\theta J_p(k)]}{k + \frac{p+1/2}{\cosh\theta}} + (\theta\to-\theta). \quad (3i)$$

Note that the counterclockwise contour must only include the zeros of equations (3c) and (3d) such that the appropriate summation over eigenvalues results. The poles at $k = (p+1/2)/\cosh\theta$ should lie *outside* the contour because they have been introduced by the normalization integral and need *not* be summed over. The situation is similar to the analysis for radial smearing functions, see [17] for more details. This observation is important because when shifting the contour towards the imaginary axis *additional* contributions result. Using the index p for all Bessel functions an intermediate result reads

$$\eta(s;\tilde{\gamma},P,\mathcal{B}_\theta) = \frac{1}{2\pi\mathrm{i}\cosh\theta}\cos\left(\frac{\pi s}{2}\right)\sum_{n=0}^{\infty}d_n(m)$$
$$\times\int_0^\infty \mathrm{d}z z^{-s}\frac{\frac{\mathrm{d}}{\mathrm{d}z}\ln\left[I_p'(z) - \frac{p}{z}I_p(z) - \mathrm{i}\,\mathrm{e}^\theta I_p(z)\right]}{\mathrm{i}z + \frac{p+1/2}{\cosh\theta}} + \frac{1}{2\pi\mathrm{i}\cosh\theta}\cos\left(\frac{\pi s}{2}\right)$$
$$\times\sum_{n=0}^{\infty}d_n(m)\int_0^\infty \mathrm{d}z z^{-s}\frac{\frac{\mathrm{d}}{\mathrm{d}z}\ln\left[I_p'(z) - \frac{p}{z}I_p(z) + \mathrm{i}\mathrm{e}^\theta I_p(z)\right]}{\mathrm{i}z - \frac{p+1/2}{\cosh\theta}}$$
$$+\frac{1}{2\cosh\theta}\sum_{n=0}^{\infty}d_n(m)\left(\frac{(p+1/2)}{\cosh\theta}\right)^{-s}$$
$$\times\ln\frac{\mathrm{d}}{\mathrm{d}k}\left[J_p'(k) + \left(\mathrm{e}^\theta - \frac{p}{k}\right)J_p(k)\right]\Big|_{k=\frac{p+1/2}{\cosh\theta}} + (\theta\to-\theta). \quad (3j)$$

The last contribution resulting from the shifting of the contour can be given in the closed form by using the differential equation for the Bessel function [25]

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}z^2}+\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}+\left(1-\frac{\nu^2}{z^2}\right)\right]J_\nu(z)=0.$$

We calculate

$$\frac{\mathrm{d}}{\mathrm{d}k}\ln\left(J_p'(k)+\left(\mathrm{e}^\theta-\frac{p}{k}\right)J_p(k)\right)\bigg|_{k=\frac{p+1/2}{\cosh\theta}}=\frac{J_p''(k)+\frac{p}{k^2}J_p(k)+\left(\mathrm{e}^\theta-\frac{p}{k}\right)J_p'(k)}{J_p'(k)+\left(\mathrm{e}^\theta-\frac{p}{k}\right)J_p(k)}\Bigg|_{k=\frac{p+1/2}{\cosh\theta}}$$
$$=\frac{J_p'(k)\left(\mathrm{e}^\theta-\frac{p+1}{k}\right)+J_p(k)\left(\frac{p(p+1)}{k^2}-1\right)}{J_p'(k)+\left(\mathrm{e}^\theta-\frac{p}{k}\right)J_p(k)}\Bigg|_{k=\frac{p+1/2}{\cosh\theta}}=\sinh\theta-\frac{\cosh\theta}{2p+1}.$$

Adding the contributions from θ and $-\theta$ the $\sinh\theta$ terms cancel and the summation over n leads to $\zeta_{\mathcal{B}}(s+1,(m-1)/2)$, which has no pole at $s=m-1$. Therefore, for the present purpose this term is irrelevant.

In the remaining integrals in (3*j*) we need, as before, only the leading term in the Olver–Debye asymptotic expansion of Bessel functions. Explicitly, with $x=1/\cosh\theta$, we obtain to leading order

$$A_0(s;\tilde{\gamma})=-\frac{\cos\left(\frac{\pi s}{2}\right)}{\pi\cosh\theta}\sum_{n=0}^{\infty}d_n(m)p^{-s}\int_0^\infty \mathrm{d}k\,k^{-s-1}\sqrt{1+k^2}\left[\frac{1}{k-\mathrm{i}x}+\frac{1}{k+\mathrm{i}x}\right]$$
$$=-\frac{2\cos\left(\frac{\pi s}{2}\right)}{\pi\cosh\theta}\sum_{n=0}^{\infty}d_n(m)p^{-s}\int_0^\infty \mathrm{d}k\frac{k^{-s}\sqrt{1+k^2}}{k^2+x^2}.$$

The k-integral is [25]

$$\int_0^\infty \mathrm{d}k\frac{k^{-s}\sqrt{1+k^2}}{k^2+x^2}=\frac{1}{2x^2}\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\,{}_2F_1\left(1,\frac{1-s}{2};\frac{1}{2};1-\frac{1}{x^2}\right),$$

and hence

$$A_0(s;\tilde{\gamma})=-\frac{1}{2}\mathrm{d}_s\frac{\cosh\theta}{\sqrt{\pi}}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)}\,{}_2F_1\left(1,\frac{1-s}{2};\frac{1}{2};-\sinh^2\theta\right)\zeta_{\mathcal{B}}\left(s,\frac{m}{2}-1\right).$$

The residue is easily evaluated and via (3*e*) we compare it with the form given in lemma 3.3 to read

$$d_2(\theta,m)=-\frac{1}{2}\cosh\theta\,{}_2F_1\left(1,1-\frac{m}{2};\frac{1}{2};-\sinh^2\theta\right),\tag{3k}$$

which implies

$$c_5(\theta,m)=\cosh\theta\,{}_2F_1\left(1,1-\frac{m}{2};\frac{1}{2};-\sinh^2\theta\right).$$

Remark 3.5. In the calculation just described it is the argument $-\sinh^2\theta$ that occurs naturally in the hypergeometric functions. Instead, the constant $c_2(\theta,m)$ in (2*d*) and $c_7(\theta,m)$ in (2*f*) have been given using $\tanh^2\theta$. In order to provide answers in a unified way one might use the transformation formula [25]

$${}_2F_1(\alpha,\beta;\gamma;z)=(1-z)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{z}{z-1}\right)$$

to write

$$c_2(\theta, m) = \frac{1}{2(m-1)}\left\{\frac{2m-5}{3} + (2-m)\cosh^2\theta\, {}_2F_1\left(1, 2-\frac{m}{2}; \frac{3}{2}; -\sinh^2\theta\right)\right\},$$

$$c_7(\theta, m) = -\frac{1}{2}\left\{1 - \cosh^2\theta\, {}_2F_1\left(1, 2-\frac{m}{2}; \frac{3}{2}; -\sinh^2\theta\right)\right\}.$$

Remark 3.6. Note that, despite the complicated appearance of the universal constants, for each specific dimension m a simple function of m and θ results. In particular, whenever the second argument of ${}_2F_1$ is 0 or a negative integer, the hypergeometric function reduces to a finite polynomial in $\sinh^2\theta$.

In order to find the missing multiplier $c_3(\theta, m)$ we present a calculation on the cylinder. In order to summarize previous results [5] we need to provide some notation. Let $M = \mathbb{R}_+ \times N$ be an even dimensional cylinder equipped with the metric $\mathrm{d}s^2 = \mathrm{d}x_m^2 + \mathrm{d}s_N^2$, where x_m is the coordinate in $\mathbb{R}_+$ and plays the role of the normal coordinate, and $\mathrm{d}s_N^2$ is the metric of the closed boundary N. The coordinates on N are denoted by $y = (y_1, y_2, \ldots, y_{m-1})$. To write down the heat kernel on M for $P^2 = (\gamma_i\nabla_i)^2$ with boundary condition $\mathcal{B}_\theta$, we call $\phi_\omega(y)$ the eigenspinors of the operator $B = \tilde{\gamma}\gamma_m\gamma_a\nabla_a$, corresponding to the eigenvalue ω, normalized so that

$$\sum_\omega \phi_\omega^\star(y)\phi_\omega(y') = \delta^{m-1}(y-y'),$$

with δ^{m-1} being the Dirac delta function, and

$$\int_N \mathrm{d}y\, \phi_\omega^*(y)\phi_\omega(y) = 1.$$

Finally we need $x = (y, x_m)$, $\xi = x_m - x_m'$, $\eta = x_m + x_m'$, $u_\omega(\eta, t) = \frac{\eta}{\sqrt{4t}} - \sqrt{t}\omega\tanh\theta$, and the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{d}\xi\, \mathrm{e}^{-\xi^2}.$$

We then have [5]

$$U(x, x'; t) = \frac{1}{\sqrt{4\pi t}}\sum_\omega \phi_\omega^*(y')\phi_\omega(y)\mathrm{e}^{-\omega^2 t}\left\{\left(\mathrm{e}^{\frac{-\xi^2}{4t}} - \mathrm{e}^{\frac{-\eta^2}{4t}}\right)\mathbf{1} + \frac{2\Pi_+\Pi_+^\star}{\cosh^2(\theta)}\left[1 + \sqrt{(\pi t)}\omega\tanh\theta e^{u_\omega^2(\eta,t)}\mathrm{erfc}(u_\omega(\eta, t))\right]\mathrm{e}^{\frac{-\eta^2}{4t}}\right\}. \quad (3l)$$

(Note that although the formal appearance of the heat kernel is identical to the one in [5], equation (5.1), the meaning of Π_+ is slightly different. The reason is that [5] considers the boundary condition resulting from, in our present notation, Π_+ whereas we consider the one resulting from Π_-. Formally the transition is obtained by reversing the sign of the normal and by using our present notation for $\Pi_\pm$.) As remarked in [5] the first term is the heat kernel on the manifold $\mathbb{R} \times N$, which does not encode any information about the boundary contribution. In the following, without changing the notation, we will ignore this term and we will determine the boundary contributions to the eta invariant from the remaining terms.

Let $f \in C^\infty(\mathrm{End}(V))$, then we want to consider $\mathrm{Tr}_{L^2}(f[P_x U(x, x'; t)]_{x=x'})$; note that the derivatives need to be performed before the coincidence limit $x = x'$ is taken. Given we have the *local* form of the heat kernel we can in principle deal with an arbitrary f. For our present purpose it is easiest to assume $f = f(y)$ only such that the x_m-integration can be done without complication.

It is natural to introduce the heat kernel $U_B(y, y'; t)$ of the operator B^2,

$$U_B(y, y'; t) = \sum_\omega \phi_\omega^*(y')\phi_\omega(y)\,\mathrm{e}^{-\omega^2 t};$$

furthermore, to make the single steps easier to follow we use the splitting

$$U_1(x, x'; t) = -\frac{1}{\sqrt{4\pi t}} \sum_\omega \phi_\omega^*(y')\phi_\omega(y)\,\mathrm{e}^{-\omega^2 t}\,\mathrm{e}^{\frac{-\eta^2}{4t}},$$

$$U_2(x, x'; t) = \frac{1}{\sqrt{4\pi t}} \sum_\omega \phi_\omega^*(y')\phi_\omega(y)\,\mathrm{e}^{-\omega^2 t}\,\frac{2\Pi_+\Pi_+^\star}{\cosh^2(\theta)} \times \left[1 + \sqrt{(\pi t)}\omega \tanh\theta\, e^{u_\omega^2(\eta,t)}\,\mathrm{erfc}(u_\omega(\eta, t))\right]\mathrm{e}^{\frac{-\eta^2}{4t}}.$$

Acting with P and performing the x_m-integration, intermediate results are

$$\int_0^\infty \mathrm{d}x_m\, f[P_x U_1(y, y', x_m, x_m'; t)]_{x_m=x_m'} = \frac{1}{\sqrt{4\pi t}}\frac{1}{2} f\gamma_m U_B(y, y'; t) - \frac{1}{4} f\gamma_m\tilde{\gamma} B_y U_B(y, y'; t), \tag{3m}$$

$$\int_0^\infty \mathrm{d}x_m\, f[P_x U_2(y, y', x_m, x_m'; t)]_{x_m=x_m'} = -\frac{1}{2\cosh^2\theta} f\gamma_m U_B(y, y'; t)\Pi_+\Pi_+^* \times \left[\frac{1}{\sqrt{\pi t}} + \omega\tanh\theta\,\mathrm{e}^{t\omega^2\tanh^2\theta}\,\mathrm{erfc}(-\sqrt{t}\omega\tanh\theta)\right] + \frac{1}{2\cosh^2\theta} f\gamma_m\tilde{\gamma} U_B(y, y'; t)\Pi_+\Pi_+^* e^{t\omega^2\tanh^2\theta}\,\mathrm{erfc}(-\sqrt{t}\omega\tanh\theta). \tag{3n}$$

Here, we have used the relation

$$-\frac{1}{2}\frac{\partial}{\partial x_m}\left[\mathrm{e}^{-x_m^2/t + u_\omega^2(2x_m,t)}\,\mathrm{erfc}(u_\omega(2x_m, t))\right] = \mathrm{e}^{-x_m^2/t}\left[\frac{1}{\sqrt{\pi t}} + \omega\tanh\theta\,\mathrm{e}^{u_\omega^2(2x_m,t)}\,\mathrm{erfc}(u_\omega(2x_m, t))\right].$$

Whereas the asymptotic $t \to 0$ behaviour in (3*m*) could be easily found from the corresponding (known) behaviour of the trace of U_B, the same is not as simple for the result in (3*n*). We have found it most convenient to perform the $L^2(N)$-trace and to relate the above equations to the zeta and eta function via

$$\zeta(s; f, P^2, \mathcal{B}_\theta) = \mathrm{Tr}_{L^2}(f(P^2)^{-s}) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{d}t\,t^{s-1}\,\mathrm{Tr}_{L^2}\big(f\,\mathrm{e}^{-tP^2}\big),$$

$$\eta(s; f, P, \mathcal{B}_\theta) = \mathrm{Tr}_{L^2}(fP(P^2)^{-s}) = \frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty \mathrm{d}t\,t^{\frac{s-1}{2}}\,\mathrm{Tr}_{L^2}\big(fP\,\mathrm{e}^{-tP^2}\big),$$

and to evaluate the asymptotic $t \to 0$ expansion from (3*e*) and

$$\mathrm{Res}\,\zeta(z; f, B^2) = \frac{a^\zeta_{\frac{m-1}{2}-z}(f, B^2)}{\Gamma(z)}. \tag{3o}$$

For (3*m*) the associated relation is readily found,

$$\eta_1(s; f, P, \mathcal{B}_\theta) = \frac{1}{\sqrt{4\pi}}\frac{1}{2}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\zeta\left(\frac{s}{2}; f\gamma_m, B^2\right) - \frac{1}{4}\eta(s; f\gamma_m\tilde{\gamma}, B).$$

In order to proceed with (3n) we note first that

$$t\omega^2 \tanh^2\theta - t\omega^2 = -\frac{t\omega^2}{\cosh^2\theta}, \qquad \operatorname{erfc}(-\sqrt{t}\omega\tanh\theta) = 1 + \operatorname{erf}(\sqrt{t}\omega\tanh\theta),$$

with the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}t\, \mathrm{e}^{-t^2}.$$

The resulting t-integral then is

$$\int_0^\infty \mathrm{d}t\, t^{\frac{s-1}{2}}\, \mathrm{e}^{-\frac{t\omega^2}{\cosh^2\theta}}(1 + \operatorname{erf}(\sqrt{t}\omega\tanh\theta)) = \frac{\cosh^{s+1}\theta}{|\omega|^{s+1}}\left[\Gamma\left(\frac{s+1}{2}\right) + \frac{2}{\sqrt{\pi}}\Gamma\left(1+\frac{s}{2}\right)\right.$$
$$\left.\times \sinh\theta\, \operatorname{sgn}(\omega)\, {}_2F_1\left(\frac{1}{2}, 1+\frac{s}{2}; \frac{3}{2}; -\sinh^2\theta\right)\right].$$

This produces the following contributions to the eta function:

$$\begin{aligned}\eta_2(s; f, P, \mathcal{B}_\theta) = &-\frac{\Gamma\left(\frac{s}{2}\right)}{2\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\cosh^2\theta}\zeta\left(\frac{s}{2}; \Pi_+\Pi_+^* f\gamma_m, B^2\right)\\
&- \frac{1}{2}\sinh\theta\cosh^{s-2}\theta\,\eta(s; \Pi_+\Pi_+^* f\gamma_m, B)\\
&- \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}\sinh^2\theta\cosh^{s-2}\theta\, {}_2F_1\left(\frac{1}{2}, 1+\frac{s}{2}; \frac{3}{2}; -\sinh^2\theta\right)\\
&\zeta\left(\frac{s}{2}; \Pi_+\Pi_+^* f\gamma_m, B^2\right) + \frac{1}{2}\cosh^{s-1}\theta\,\eta(s; \Pi_+\Pi_+^* f\gamma_m\tilde{\gamma}; B)\\
&+ \frac{\cosh^{s-1}\theta\sinh\theta\,\Gamma\left(1+\frac{s}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)}\, {}_2F_1\left(\frac{1}{2}; 1+\frac{s}{2}; \frac{3}{2}; -\sinh^2\theta\right)\\
&\zeta\left(\frac{s}{2}; \Pi_+\Pi_+^* f\gamma_m\tilde{\gamma}, B^2\right).\end{aligned}$$

From here, with the help of (3o) and (3e), it is easy to find the residue of $\eta(s; f, P, \mathcal{B}_\theta)$ at $s = m-1$, needed for the evaluation of $a_1^\eta(f, P, \mathcal{B}_\theta)$. We find

$$\begin{aligned}\operatorname{Res}\eta(m-1; f, P, \mathcal{B}_\theta) = \frac{1}{\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)}&\left\{\frac{1}{2}a_0(f\gamma_m, B^2) - \frac{1}{\cosh^2\theta}a_0(\Pi_+\Pi_+^* f\gamma_m, B^2)\right.\\
&+ (m-1)\sinh\theta\cosh^{m-2}\theta\, {}_2F_1\left(\frac{1}{2}, \frac{m+1}{2}; \frac{3}{2}; -\sinh^2\theta\right)\\
&\left.\times\left[a_0(\Pi_+\Pi_+^* f\gamma_m\tilde{\gamma}, B^2) - \tanh\theta\, a_0(\Pi_+\Pi_+^* f\gamma_m, B^2)\right]\right\}.\end{aligned}$$

The leading heat kernel coefficient $a_0(G, B^2)$ is of course known for an arbitrary endomorphism G; it is

$$a_0(G, B^2) = (4\pi)^{-\frac{m-1}{2}}\int_N \mathrm{d}y\, \operatorname{Tr}_V(G).$$

In order to obtain the invariant form given in lemma 3.3, we evaluate $\Pi_+\Pi_+^*$ in the form

$$\Pi_+\Pi_+^* = \tfrac{1}{2}\cosh\theta(\cosh\theta + \tilde{\gamma}\sinh\theta - \tilde{\gamma}\gamma_m).$$

Adding up all pieces this shows

$$\operatorname{Res}\eta(m-1;f,P,\mathcal{B}_\theta)=\frac{2}{\Gamma\left(\frac{m}{2}\right)}(4\pi)^{-m/2}\int_N \mathrm{d}y\operatorname{Tr}_V\left\{f\gamma_m\cdot 0\right.$$
$$+\ f\gamma_m\tilde{\gamma}\left[\frac{1}{2}\tanh\theta-\frac{1}{2}(m-1)\sinh\theta\cosh^{m-2}\theta\,{}_2F_1\left(\frac{1}{2},\frac{m+1}{2};\frac{3}{2};-\sinh^2\theta\right)\right]$$
$$-\ f\frac{1}{2}(m-1)\sinh\theta\cosh^{m-1}\theta\,{}_2F_1\left(\frac{1}{2},\frac{m+1}{2};\frac{3}{2};-\sinh^2\theta\right)$$
$$\left.-\ f\tilde{\gamma}\left[\frac{1}{2\cosh\theta}+\frac{1}{2}(m-1)\sinh^2\theta\cosh^{m-2}\theta\,{}_2F_1\left(\frac{1}{2},\frac{m+1}{2};\frac{3}{2};-\sinh^2\theta\right)\right]\right\}.$$

From this result we can read off $d_i(\theta,m)$, $i=1,\ldots,4$; we find

$$d_1(\theta,m)=-\frac{m-1}{2}\sinh\theta\cosh^{m-1}\theta\,{}_2F_1\left(\frac{1}{2},\frac{m+1}{2};\frac{3}{2};-\sinh^2\theta\right),$$
$$d_2(\theta,m)=-\frac{1}{2\cosh\theta}-\frac{m-1}{2}\sinh^2\theta\cosh^{m-2}\theta\,{}_2F_1\left(\frac{1}{2},\frac{m+1}{2};\frac{3}{2};-\sinh^2\theta\right),$$
$$d_3(\theta,m)=0,$$
$$d_4(\theta,m)=-\frac{1}{2}\tanh\theta+\frac{m-1}{2}\sinh\theta\cosh^{m-2}\theta\,{}_2F_1\left(\frac{1}{2},\frac{m+1}{2};\frac{3}{2};-\sinh^2\theta\right).$$

The result for $d_1(\theta,m)$ can be seen to agree with the result on the ball, equation (3*g*), by using the transformation formula ([25], equation (9.131.1))

$$ {}_2F_1(\alpha,\beta;\gamma;z)=(1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z). \tag{3p}$$

In order to show that the results for $d_2(\theta,m)$ coming from the ball and cylinder agree, we need to show that

$$\cosh^2\theta\,{}_2F_1\left(1,1-\frac{m}{2};\frac{1}{2};-\sinh^2\theta\right)$$
$$=1+(m-1)\sinh^2\theta\cosh^{m-1}\theta\,{}_2F_1\left(\frac{1}{2},\frac{m+1}{2};\frac{3}{2};-\sinh^2\theta\right). \tag{3q}$$

To see this, we first apply the above transformation formula, equation (3*p*), and then the Gauss recursion formula ([25], equation (9.137.12))

$$\gamma\,{}_2F_1(\alpha,\beta;\gamma;z)-\gamma\,{}_2F_1(\alpha+1,\beta;\gamma;z)+\beta z\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z)=0$$

with $\alpha=-1/2$, $\beta=(m-1)/2$, $\gamma=1/2$ and $z=-\sinh^2\theta$. Thus, all results obtained are consistent and we have determined the full a_1 and a_2 coefficient for chiral bag boundary conditions.

4. Concluding remarks

For the case of operators P of Dirac type subject to local boundary conditions of chiral bag type as in equation (1*b*), we have studied the asymptotic expansion as $t\to 0^+$ of the smeared L^2-trace of the associated heat semigroup, i.e.

$$\operatorname{Tr}_{L^2}\left(fe^{-tP^2}\right)\sim\sum_{n=0}^{\infty}t^{(n-m)/2}a_n(f,P^2,\mathcal{B}_\theta). \tag{4.1}$$

On using functorial methods, special case calculations and the relation between η- and ζ-invariants, we have succeeded in evaluating the full boundary contribution to the a_1 and

a_2 coefficients, the functional form of which is given by equations (2*a*) and (2*b*). Our contributions are of technical but non-trivial nature, because both functorial methods and the theory of the η-invariant require a lot of work to obtain the desired a_2 coefficient. It now appears possible that, by exploiting the methods described in our paper, further heat-kernel coefficients will be obtained, if they are needed in physical or mathematical applications. In turn, a better understanding of the spectral functions of modern mathematical physics [27] will also be gained.

Acknowledgments

Research of GE was partially supported by the INFN (Naples, Italy) and by PRIN 2002 *SINTESI*. Research of PG was partially supported by the MPI (Leipzig, Germany). KK acknowledges support by the Baylor University Summer Sabbatical Program, by the MPI (Leipzig, Germany), and by the INFN (Naples, Italy) and PRIN 2002 *SINTESI*.

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